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THE PENNSYLVANIA STATE UNIVERSITY

IONOSPHERIC RESEARCH

Scientific Report No. 178

PERTURBATION METHODS APPLIED TO THE REFLECTION OF RADIO WAVES FROM THE IONOSPHERE

by

P. W. Norman

February 1, 1963

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IONOSPHERE RESEARCH LABORATORY

University Park, Pennsylvania

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IONOSPHERE RESEARCH LABORATORY

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Abstract

A perturbation approach to the problem of determining the electromagnetic field reflected from the ionosphere is presented. It is shown that by treating the earth's field of magnetic induction as a perturbation, Maxwell's equations and the constitutive relationship are considerably simplified.

First order approximations for the field reflected from a homogeneous sharply bounded ionosphere are found.

The perturbation method to be developed has certain limitations which are discussed.

I. INTRODUCTION

I.I. The Reflection Problem Formulated

The problem of electromagnetic reflection from the ionosphere will now be formulated in very general terms.

The coordinate system is arranged such that the z-axis is vertical and the earth's field of magnetic induction lies in the x-z plane. The ionosphere is assumed to be horizontally stratified, non-magnetic, and confined to the half space $z > 0$. The region $z < 0$ is assumed to be free space. It is also assumed that a known electromagnetic source is located in the free space region $z < 0$. Let the source, and therefore the field quantities, have a harmonic time variation of the form $e^{-i\omega t}$. Since the ionosphere has been assumed horizontally stratified, the electron density and collision frequency will, in general, be a function of the coordinate z. These functions are assumed to be at least piece-wise continuous.

For the purposes of ionospheric wave propagation, Maxwell's Equations (MKS units) and the constitutive relationship may be written[1]

$$\nabla \times \overline{E} = i\omega\mu_0 \overline{H} \tag{1}$$

$$\nabla \times \overline{H} = i\omega\varepsilon_0 \left(\overline{E} + \frac{\overline{P}}{\varepsilon_0}\right) + \overline{J} \tag{2}$$

$$\frac{U(Y^2 - U^2)}{X} \frac{\overline{P}}{\varepsilon_0} = U^2\overline{E} - \overline{Y}(\overline{Y}\cdot\overline{E}) - iU\,\overline{Y} \times \overline{E} \tag{3}$$

$$X = \frac{Ne^2}{m\varepsilon_0\omega^2} \qquad \overline{Y} = \frac{e}{m\omega}\overline{B} \qquad U = 1 + i\nu/\omega \,.$$

where

N = Electron density

e = Electronic charge

m = Electronic mass

$\overline{B}$ = Earth's field of magnetic induction

ν = Collision frequency

j = Current density of the source

ω = Angular frequency .

The plane $z = 0$ is an interface between two media; namely the ionosphere and free space. If $\overline{j} = 0$ on the plane $z = 0$, then the tangential components of $\overline{E}$ and $\overline{H}$ must be continuous on the plane $z = 0$.

The problem statement is extremely simple. Given the current density of the source and an ionospheric model, find $\overline{E}$ and $\overline{H}$ such that equations (1) and (2) are satisfied and the tangential components of E and H are continuous on the plane $z = 0$.

I.II General Remarks

Without loss of generality, the source may be replaced by a point dipole. The dipole field may be represented as an integral over plane waves[2,3].

A possible method of attack would be to find the field reflected from the ionosphere when a plane wave is incident at an arbitrary angle upon the ionosphere. Then, by integration over the incident angles, the reflected field of a point dipole may be obtained. But first a plane wave solution must be found.

I.III Problem of Obtaining a Plane Wave Solution

A great deal of work has been done with plane waves in the ionosphere. However, most methods are not of sufficient generality to be of use in the solution of this problem. It seems three basic methods of finding solutions for a plane wave of arbitrary incidence are available; other methods are seemingly special cases of these three. These methods are listed below:

1) Booker Quartic[1]

The ionosphere is first assumed homogeneous. The roots of the Booker Quartic give the propagation constants. Two of the roots are associated with the up-going plane waves and the other two roots are associated with down-going plane waves. The solution is extended to an inhomogeneous ionosphere by using the W.K.B. approximating technique.

2) First order coupled equations[4]

Clemmow and Heading present a set of first order coupled equations for the quantities E_x, E_y, H_x and H_y. The equations uncouple in a homogeneous medium. Keller[5] gives a detailed method of solving these equations.

3) Integral equation variational technique[6]

A variational technique is presented for the reflection and transmission matrices. The power of this technique is that the detailed solution of the fields in the ionosphere never need be found.

The methods mentioned above have the disadvantage of being rather laborious. Also the solutions obtained by these methods involved the angles of incident in a complicated fashion. Hence, the integrals over the resulting plane waves cannot be evaluated short of numerical means.

II. DEVELOPMENT OF PERTURBATION METHOD

Maxwell's Equations and the constitutive relationship are written

$$\nabla \times \overline{E} = i\omega\mu_o \overline{H} \tag{1}$$

$$\nabla \times \overline{H} + i\omega\varepsilon_o \left(\overline{E} + \frac{\overline{P}}{\varepsilon_o}\right) = \overline{J} \tag{2}$$

$$\frac{U(Y^2-U^2)}{X} \frac{\overline{P}}{\varepsilon_o} = U^2\overline{E} - \overline{Y}(\overline{Y}\cdot\overline{E}) - iU\,\overline{Y} \times \overline{E} \,. \tag{3}$$

Treat the parameter $|\overline{Y}|$ as a perturbation parameter. Since $|\overline{Y}|$ is directly proportional to the earth's field of magnetic induction $|\overline{B}|$, treating $|\overline{Y}|$ as a perturbation is equivalent to treating $|\overline{B}|$ as a perturbation.

Expanded $\overline{E}$, $\overline{H}$, and $\overline{P}$ is a power series in $|\overline{Y}|$ about $|\overline{Y}| = 0$

$$\overline{E} = \sum_{n=0}^{\infty} |\overline{Y}|^n \overline{E}^n$$

$$\overline{H} = \sum_{n=0}^{\infty} |\overline{Y}|^n \overline{H}^n$$

$$\overline{P} = \sum_{n=0}^{\infty} |\overline{Y}|^n \overline{P}^n \quad . \tag{4}$$

The expansions (4) are substituted into equations (1), (2), and (3) and the vector coefficients of like powers of $|\overline{Y}|$ are equated to obtain the following relations:

$$\nabla \times \overline{E}^n = i\omega\mu_o \overline{H}^n \tag{5}$$

$$\nabla \times \overline{H}^n + i\omega\varepsilon_o \left(\overline{E}^n + \frac{\overline{P}^n}{\varepsilon_o}\right) = \delta_{no}\overline{j} \qquad n \geqslant 0 \tag{6}$$

$$-\frac{U}{X}\frac{\overline{P}^o}{\varepsilon_o} = \overline{E}^o \tag{7}$$

$$-\frac{U}{X}\frac{\overline{P}^1}{\varepsilon_o} = \overline{E}^1 - \frac{i}{U}\,\overline{\sigma} \times \overline{E}^o \tag{8}$$

$$\frac{U}{X}\frac{\overline{P}^{n-2}}{\varepsilon_o} - \frac{U^3}{X}\frac{\overline{P}^n}{\varepsilon_o} = U^2\,\overline{E}^n - \overline{\sigma}(\overline{\sigma}\cdot\overline{E}^{n-2}) - iU\overline{\sigma} \times \overline{E}^{n-1} \qquad n \geqslant 2 \tag{9}$$

where

$$\overline{Y} = \left|\overline{Y}\right|\,\overline{\sigma} \qquad\qquad \delta_{no} = \begin{matrix} 1 & n = 0 \\ 0 & n \neq 0\,. \end{matrix}$$

Equations (7), (8), and (9) are used to eliminate $\overline{P}^n$ from equation (6). Equation (6) becomes, with $n \geqslant 0$:

$$\nabla \times \overline{H}^n + i\omega\varepsilon_o\mu^2\,\overline{E}^n = \overline{I}^n$$

where

$$\mu^2 = 1 - \frac{X}{U}$$

$$\overline{I}^o = \overline{j}$$

$$\overline{I}^1 = \frac{\omega\varepsilon_o}{U^2}\,X\,\overline{\sigma} \times \overline{E}^o$$

$$\overline{I}^n = -\frac{i\omega\varepsilon_o}{U^2}\,X\,\overline{\sigma}\,(\overline{\sigma}\cdot\overline{E}^{n-2}) + \frac{\omega\varepsilon_o}{U^2}\,X\,\overline{\sigma} \times \overline{E}^{n-1}$$
$$-\frac{i\omega\varepsilon_o}{U^2}\,\overline{P}^{n-2} \qquad n \geqslant 2\,.$$

The problem of finding the solutions of equations (1) and (2) has been reduced to finding the solution of the following infinite set of equations:

$$\nabla \times \bar{E}^n = i\omega\mu_o \bar{H}^n \tag{10}$$

$$\nabla \times \bar{H}^n + i\omega\varepsilon_o \mu^2 \bar{E}^n = \bar{I}^n \tag{11}$$

The following observations are made:

1) If $\bar{E}^n$ is known, then equation (10) may be used to determine $\bar{H}^n$. Hence, only $\bar{E}^n$ need be found.

2) $\bar{E}^n$ and $\bar{H}^n$ are bounded since $\bar{E}$ and $\bar{H}$ are bounded. If $\bar{J} = \bar{I}^o = 0$ on the plane $z = 0$, then the tangential components of $\bar{E}^n$ and $\bar{H}^n$ $(n > 0)$ must be continuous on the plane $z = 0$. The validity of this statement may be deduced from equations (10) and (11) in the usual manner.

3) $\bar{I}^n$ depends on $\bar{E}^{n-1}$ and $\bar{E}^{n-2}$. Therefore, if $\bar{E}^{n-1}$ and $\bar{E}^{n-2}$ are known, $\bar{I}^n$ is known.

4) The equations (10) and (11), which determine $\bar{E}^n$, take the form of Maxwell's equation in an isotropic medium with a source current density $\bar{I}^n$.

Equations (10) and (11) with $n = 0$ are used to find $\bar{E}^o$. Once $\bar{E}^o$ is found, $\bar{I}^1$ is known. Then equations (10) and (11) are again used but with $n = 1$ to find $\bar{E}^1$. This process can be continued to calculate as many $\bar{E}^n$'s as desired.

In summary, the problem of finding the solutions of Maxwell's equations in an anisotropic medium is reduced to finding the solution of an infinite set of equations by treating $|\bar{Y}|$ as a perturbation. The n-th equation of the set takes the form of Maxwell's equations in an isotropic medium.

When $\bar{\bar{E}}^{n-2}$ and $\bar{\bar{E}}^{n-1}$ have been determined, $\bar{I}^{n}$ becomes known, and $\bar{E}^{n}$ can be found. $\bar{E}^{n}$ is subject to the same boundary conditions as $\bar{E}$.

The physical interpretation of the perturbation series $\bar{\bar{E}} = \sum_{n=0}^{\infty} \left|\bar{Y}\right|^{n} \bar{\bar{E}}^{n}$ is interesting. $\bar{E}^{o}$ is the solution of the propagation problem with no earth's magnetic field. The terms $\left|Y\right| \bar{\bar{E}}^{1}$, $\bar{Y}^{2} E^{2}$, etc. are correction terms.

II.I Limitations of the Perturbation Approach

The convergence properties of the series (4) cannot be known until the general term is found. In most practical applications the determination of the general term would be at best laborious. However, if $\left|\bar{Y}\right| \ll 1$ the first few terms in the series should be sufficient to represent the solution accurately. The charge to mass ratio of the electron and the earth's field of magnetic induction are fixed quantities. Therefore, the only parameter in $\left|\bar{Y}\right|$ that may be varied is ω. Thus, $\left|\bar{Y}\right| \ll 1$ implies high frequencies.

As has been previously mentioned, equations (10) and (11) take the form of Maxwell's equation in an isotropic medium. However, since μ is a function of position the medium is inhomogeneous. Thus, the positional dependance of μ must be such that Maxwell's equations can be solved.

III. PERTURBATION THEORY APPLIED TO A SPECIAL CASE

The ideas developed in the last section will now be used to find first order correction terms for the field reflected from a sharply bounded homogeneous ionosphere. Let the interface between the ionosphere and free space be the plane $z = 0$. A vertical point dipole is located at $(0, 0, -d)$. The earth's field of magnetic induction is assumed to be vertical. Since the perturbation approximation is applicable only to high frequencies, collisions will be neglected.

Equations (10) and (11), with $n = 0$, are

$$\nabla \times \overline{E}^{o} = i\omega\mu_{o}\overline{H}^{o} \tag{12}$$

$$\nabla \times \overline{H}^{o} = -i\omega\varepsilon_{o}\mu^{2}\overline{E}^{o} + \overline{j} \tag{13}$$

where

$$\mu^{2} = \begin{cases} 1 - X & z > 0 \\ 1 & z < 0 \end{cases}$$

$$j = 1_{z}\, I\, \delta(\overline{r}_{o} - \overline{r})$$

$\delta(\overline{r}_{o} - \overline{r})$ = Dirac delta function.

$$\overline{r}_{o} = -1_{z}\, d$$

1_{z} = Unit vector in the z-direction .

In general, the symbol 1_{k} will mean a unit vector in the k^{th} direction.

If $\overline{H}^{o}$ is eliminated from equations (1) and (1) the result is

$$\nabla \times (\nabla \times \overline{E}^{o}) - k^{2}\overline{E}^{o} = i\omega\mu_{o}\overline{j} \tag{14}$$

$$k^2 = \omega^2\mu_o\varepsilon_o\mu^2.$$

Let

$$\bar{E}^o = \begin{cases} \bar{E}^+ & z > 0 \\ \bar{E}^- & z < 0 \end{cases} .$$

Then it follows from equation (14) that $\bar{E}^+$ and $\bar{E}^-$ must satisfy

$$\nabla \times (\nabla \times \bar{E}^+) - k^2\bar{E}^+ = 0 \tag{15}$$

$$\nabla \times (\nabla \times \bar{E}^-) - k_o^2\bar{E}^- = i\omega\mu_o\bar{J} . \tag{16}$$

Let us introduce two new vectors, $\bar{\pi}^+$ and π^-, defined by

$$\bar{E}^+ = \nabla(\nabla\cdot\bar{\pi}^+) + k^2\bar{\pi}^+ , \tag{17}$$

$$\bar{E}^- = \nabla(\nabla\cdot\bar{\pi}^-) + k_o^2\bar{\pi}^- . \tag{18}$$

Then, from equations (15) and (16),

$$(\nabla^2 + k^2)\,\bar{\pi}^+ = 0 \tag{19}$$

$$(\nabla^2 + k_o^2)\,\bar{\pi}^- = -\frac{i\omega\mu_o}{k_o^2}\bar{J} . \tag{20}$$

Since $\bar{J}$ has only a z-component, $\pi_x^{\pm} = 0$, $\pi_y^{\pm} = 0$, and $\pi_z^{\pm} = \pi^{\pm}$.

From equations (17), (18), and (12) the following expressions are found for the field components in cylindrical coordinates:

$$E_r^+ = \frac{\partial}{\partial r}\frac{\partial\pi^+}{\partial z} \qquad E_r^- = \frac{\partial}{\partial r}\frac{\partial\pi^-}{\partial z}$$

$$E_\varphi^+ = \frac{1}{r}\frac{\partial}{\partial\varphi}\frac{\partial\pi^+}{\partial z} \qquad E_\varphi^- = \frac{1}{r}\frac{\partial}{\partial\varphi}\frac{\partial\pi^-}{\partial z}$$

$$H_r^+ = \frac{k^2}{i\omega\mu_o}\frac{1}{r}\frac{\partial\pi^+}{\partial\varphi} \qquad H_r^- = \frac{k_o^2}{i\omega\mu_o}\frac{1}{r}\frac{\partial\pi^-}{\partial\varphi}$$

$$H_\theta^+ = -\frac{k^2}{i\omega\mu_o}\frac{\partial\pi^+}{\partial r} \qquad H_\theta^- = -\frac{k_o^2}{i\omega\mu_o}\frac{\partial\pi^-}{\partial r} \quad .$$

The conditions $E_r^+ = E_r^-$ and $E_\theta^+ = E_\theta^-$ evaluated at $z = 0$ imply

$$\frac{\partial}{\partial r}\frac{\partial\pi^+}{\partial z} = \frac{\partial}{\partial r}\frac{\partial\pi^-}{\partial z} \qquad z = 0 \tag{21}$$

$$\frac{\partial}{\partial\varphi}\frac{\partial\pi^+}{\partial z} = \frac{\partial}{\partial\varphi}\frac{\partial\pi^-}{\partial z} \qquad z = 0 \ . \tag{22}$$

Equations (21) and (22) may be integrated with respect to r and φ; respectively. In both cases the constants of integration must be zero since $\frac{\partial\pi^\pm}{\partial z}$ tends to zero as r tends to infinity.

In a similar manner it is found that the conditions $H_{r,\varphi}^+ = H_{r,\varphi}^-$ evaluated at $z = 0$ are satisfied if

$$n^2\pi^+ = \pi^- \qquad z = 0$$

Thus it can be concluded that if $\pi^\pm$ are such that

$$\frac{\partial\pi^+}{\partial z} = \frac{\partial\pi^-}{\partial z} \qquad z = 0 \tag{23}$$

$$n^2\pi^+ = \pi^- \qquad z = 0 \tag{24}$$

then the tangential components $\bar{E}$ and $\bar{H}$ will be continuous on the plane $z = 0$.

Let $\pi^- = \pi^P + \pi^S$

where

$$(\nabla^2 + k_o^2)\ \pi^P = -\frac{i\omega\mu_o I}{k_o^2}\ \delta(\bar{r}_o - \bar{r}) \tag{25}$$

$$(\nabla^2 + k_o^2)\,\pi^S = 0 \quad . \tag{26}$$

The solution of equation (25)[7] is

$$\pi^P = \frac{i\omega\mu_o I}{4\pi\, k_o^2}\,\frac{e^{ik_o\left|\bar{r} - \bar{r}_o\right|}}{\left|\bar{r} - \bar{r}_o\right|} \quad \text{where } \bar{r}_o = -1_z d \quad .$$

If the Sommerfeld integral representation[7] for $\frac{e^{ik_o\left|\bar{r} - \bar{r}_o\right|}}{\left|\bar{r} - \bar{r}_o\right|}$ is used, π^P may be written:

$$\pi^P = P\int_o^\infty J_o(\lambda r)e^{-\gamma|z+d|}\,\frac{\lambda d\lambda}{\gamma} \tag{27}$$

where

$$P = \frac{i\omega\mu_o I}{4\pi\, k_o^2}$$

$$\gamma = \sqrt{\lambda^2 - k_o^2}$$

$J_n(\lambda r)$ = Cylindrical Bessel function of the n^{th} order.

Equations (26) and (19) are satisfied if

$$\pi^S = P\int_o^\infty f^S(\lambda)J_o(\lambda r)e^{-\gamma|d-z|}\lambda d\lambda \qquad z \leqslant 0 \tag{28}$$

$$\pi^+ = P\int_o^\infty f^+(\lambda)e^{-\gamma d}J_o(\lambda r)e^{-\gamma_E z}\,\lambda d\lambda \quad z > 0$$

where

$$\gamma_E = \sqrt{\lambda^2 - k^2} \quad . \tag{29}$$

The functions $f^S(\lambda)$ and $f^+(\lambda)$ must be determined from the boundary conditions.

The following points should be noted:

1) The function π^P is singular at (0, 0, - d) since the integral (26) fails to converge at this point. This singularity is caused by the presence of the dipole.

2) The functions π^S and π^+ are continuous throughout the regions $z < 0$ and $z > 0$, respectively.

3) All three functions obey the Sommerfeld radiation condition at infinity.

By applying the boundary conditions expressed by equations (23) and (24) it is found that

$$f^+(\lambda) = \frac{2}{\mu^2\gamma + \gamma_E} \tag{30}$$

$$f^S(\lambda) = \frac{1}{\gamma} - \frac{2\gamma_E}{\gamma(\mu^2\gamma + \gamma_E)} \quad . \tag{31}$$

These solutions were first obtained by Sommerfeld in his 1909 paper[7].

Let $F(\lambda) = Pf^+(\lambda)\, e^{-\gamma d}\lambda$

$$\vec{\pi}^+ = 1_z \int_0^\infty F(\lambda) J_o(\lambda r) e^{-\gamma_E z}\, d\lambda \qquad z \geqslant 0 \quad .$$

Equations (10) and (11) with n = 1 are

$$\nabla \times \overline{E}^1 = i\omega\mu_o \overline{H}^1 \tag{31}$$

$$\nabla \times \overline{H}^1 = - i\omega\varepsilon_o\mu^2\overline{E}^1 + \vec{I}^1 \quad . \tag{32}$$

Eliminating $\overline{H}^1$ from equations (31) and (32) gives the result

$$\nabla \times (\nabla \times \overline{E}^{1+}) - k^2\overline{E}^{1+} = i\omega\mu_o\overline{I}^1 \tag{33}$$

$$\nabla \times (\nabla \times \overline{E}^{1-}) - k_o^2\overline{E}^{1-} = 0 \quad , \tag{34}$$

where

$$\overline{E}^1 = \begin{matrix} \overline{E}^{1+} & z > 0 \\ \overline{E}^{1-} & z < 0 \end{matrix} \quad .$$

We now introduce the vectors $\overline{\pi}^{1\pm}$ defined by

$$\overline{E}^{1+} = \nabla(\nabla\cdot\overline{\pi}^{1+}) + k^2\overline{\pi}^{1+}$$

$$\overline{E}^{1-} = \nabla(\nabla\cdot\overline{\pi}^{1-}) + k_o^2\overline{\pi}^{1-}$$

into equations (33) and (34) to obtain the following wave equations:

$$(\nabla^2 + k^2)\,\overline{\pi}^{1+} = -\frac{i\omega\mu_o}{k^2}\,\overline{I}^1 \tag{35}$$

$$(\nabla^2 + k_o^2)\,\overline{\pi}^{1-} = 0 \tag{36}$$

$$\overline{I}^1 = \omega\, e_o\, X\, \overline{\sigma}\, X\, \overline{E}^o \qquad\qquad \overline{\sigma} = 1_z$$

$$\overline{\sigma} \times \overline{E} = 1_z \times (1_r E_r^+ + 1_z E_z^+) = 1_z \times 1_r E_r^+$$

$$\overline{\sigma} \times \overline{E} = 1_\varphi E_r^+ \quad .$$

Thus equation (35) becomes

$$(\nabla^2 + k^2)\,\overline{\pi}^{1+} = -\frac{iX}{\mu^2}\; 1_\varphi E_r^+$$

with

$$E_r^+ = \frac{\partial}{\partial r}\frac{\partial \pi^+}{\partial z} \quad .$$

Let $\overline{\pi}^{1+} = \overline{\pi}^{1P} + \overline{\pi}^{1S}$,

where

$$(\nabla^2 + k^2)\ \bar{\pi}^{1P} = -\frac{iX}{\mu^2} 1_\varphi E_r^+ \tag{37}$$

$$(\nabla^2 + k^2)\ \bar{\pi}^{1S} = 0 \tag{38}$$

$$\bar{\pi}^{1P} = \frac{iX}{4\pi\mu^2} \int \frac{e^{ik|\bar{r}-\bar{r}^1|}}{|\bar{r}-\bar{r}^1|}\ 1_\varphi E_r^+ (\bar{r}^1)\ d\bar{r}^1 \tag{39}$$

$$E_r^+ = \int_0^\infty F(\lambda)\ \gamma_E\ J_1(\lambda r)\ e^{-\gamma_E z} \lambda d\lambda \quad . \tag{40}$$

The integral in equation (39) may be evaluated by using the following facts [8,9]:

1. $$\frac{e^{ik|\bar{r}-\bar{r}^1|}}{|\bar{r}-\bar{r}^1|} = \sum_{m=0} \varepsilon_m \cos m\ (\varphi-\varphi^1) \int_0^\infty J_m(\lambda r) J_m(\lambda r^1)\ x\ \frac{e^{-\gamma_E|z-z^1|}}{\gamma_E}\ \lambda d\lambda$$

where

$$\varepsilon_m = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix} \begin{matrix} m = 0 \\ m \neq 0 \end{matrix} \quad .$$

2. $$\int_0^\infty J_m(\lambda r^1) J_m(\lambda^1 r^1) r^1\ dr^1 = \frac{\delta(\lambda-\lambda^1)}{\sqrt{\lambda\lambda^1}} \quad .$$

3. $1_\varphi = -1_x \sin\varphi + 1_y \cos\varphi$.

$$\bar{\pi}^{1P} = \frac{iX}{2\mu^2} 1\varphi \int_0^\infty F(\lambda)\ J_1(\lambda r) q(z) e^{-\gamma_E z} \lambda d\lambda \tag{41}$$

where

$$q(z) = z + \frac{1}{2\gamma_E}$$

$$\pi_\varphi^{1P} = 1_\varphi \cdot \bar{\pi}^{1P} = \frac{iX}{2\mu^2} \int_0^\infty F(\lambda)\ J_1\ (\lambda r)\ q\ (z)\ e^{-\gamma_E z}\ \lambda d\lambda \quad . \tag{42}$$

It will be found that in order to satisfy the boundary conditions, $\bar{\pi}^{1S}$ and $\bar{\pi}^{1-}$ will have to have a z-component as well as a φ-component

$$\bar{\pi}^{1+} = (0,\ \pi_\varphi^{1P} + \pi_\varphi^{1S},\ \pi_z^{1S})\ ,$$

$$\bar{\pi}^{1-} = (0,\ \pi_\varphi^{1-},\ \pi_z^{1-})\ .$$

The tangential components of the fields are:

$$E_r^{1\pm} = \frac{\partial}{\partial r}\nabla\cdot\bar{\pi}^{1\pm} \tag{43}$$

$$E_\varphi^{1+} = \frac{1}{r}\frac{\partial}{\partial\theta}\nabla\cdot\bar{\pi}^{1\pm} + k^2\pi_\varphi^{1\pm} \tag{44}$$

$$H_r^{1\pm} = \frac{k^2}{i\omega\mu_o}\left(\frac{1}{r}\frac{\partial\pi_z^{1\pm}}{\partial\varphi} - \frac{\partial\pi_\varphi^{1\pm}}{\partial z}\right) \tag{45}$$

$$H_\varphi^{1\pm} = -\frac{k^2}{i\omega\mu_o}\frac{\partial\pi_z^{1\pm}}{\partial r}\ . \tag{46}$$

The condition $E_r^{1+} = E_r^{1-}$ evaluated at $z = 0$ implies

$$\frac{\partial}{\partial r}\nabla\cdot\bar{\pi}^{\pm} = \frac{\partial}{\partial r}\nabla.\bar{\pi}^{1-} \qquad z = 0\ . \tag{47}$$

Equation (47) is integrated with respect to r to obtain

$$\nabla\cdot\bar{\pi}^{1+} = \nabla\cdot\bar{\pi}^{1-} \qquad z = 0\ . \tag{48}$$

The constant of integration is zero since $\nabla\cdot\bar{\pi}^{1\pm}$ tends to zero as r tends to infinity.

By using the continuity of E_φ^1, equations (44) and (48), it is found that

$$\mu^2\pi_\varphi^{1+} = \pi_\varphi^{1-}\ , \qquad z = 0\ . \tag{49}$$

The continuity of H_φ^{1+} and equation (46) implies

$$\mu^2 \pi_z^{1+} = \pi_z^{1-} , \qquad z = 0 . \qquad (50)$$

Finally, by using equations (45) and (50) and the continuity of H_r^1 , it is found that

$$\mu^2 \frac{\partial \pi_\varphi^{1+}}{\partial z} = \frac{\partial \pi_\varphi^{1-}}{\partial z} , \qquad z = 0 . \qquad (51)$$

The following two boundary conditions are used to determine the arbitrary functions that will appear in the integral representation of $\pi_\varphi^{1\pm}$:

$$\left.\begin{aligned} \mu^2 \pi_\varphi^{1+} &= \pi_\varphi^{1-} \\ \mu^2 \frac{\partial \pi_\varphi^{1+}}{\partial z} &= \frac{\partial \pi_\varphi^{1-}}{\partial z} \end{aligned}\right\} , \qquad z = 0 . \qquad (52)$$

When $\pi_\varphi^{\pm}$ has been found, the two boundary conditions

$$\left.\begin{aligned} \nabla \cdot \bar{\pi}^{1+} &= \nabla \cdot \bar{\pi}^{1-} \\ \mu^2 \bar{\pi}_z^{1+} &= \pi_z^{1-} \end{aligned}\right\} , \qquad z = 0 \qquad (53)$$

are used to find the arbitrary functions that will appear in the integral representations of $\pi_z^{1\pm}$.

It may be verified by direct substitution that the relations

$$\pi_\varphi^{1S} = \frac{iX}{2\mu^2} \int_0^\infty G^S(\lambda) J_1(\lambda r)\ e^{-\gamma_E z}\ \lambda d\lambda, \quad z > 0$$

$$\pi_\varphi^{1-} = \frac{iX}{2\mu^2} \int_0^\infty G^-(\lambda) J_1(\lambda r)\ e^{\gamma z}\ \lambda d\lambda, \quad |d| > |z| \quad z < 0$$

$$\pi_z^{+1} = \frac{iX}{2\mu^2}\int_0^\infty \Phi^+(\lambda)J_0(\lambda r)\ e^{-\gamma_E z}\lambda d\lambda,\ z>0$$

$$\pi_z^{-1} = \frac{iX}{2\mu^2}\int_0^\infty \Phi^-(\lambda)J_0(\lambda r)\ e^{-\gamma z}\lambda d\lambda, |d|>|z| \quad z<0$$

satisfy equations (38) and (36) if $G^S(\lambda)$, $G^-(\lambda)$, $\Phi^+(\lambda)$, and $\Phi^-(\lambda)$ are integrable functions.

The boundary conditions (52) and (53) are used to find:

$$\Phi^+ = \Phi^- = 0$$

$$G^-(\lambda) = \frac{\mu^2 F(\lambda)}{\gamma+\gamma_E}$$

$$G^+(\lambda) = \left[\frac{1}{\gamma+\gamma_E} - \frac{1}{2\gamma_E}\right] F(\lambda) \quad .$$

Since the fields in the region $|d| \geqslant |z|$ $z<0$ are the observable quantities, the fields in the region $z>0$ will not be considered further, except to mention that $\lim\limits_{X\to 1} \overline{\pi}^{1+} = \infty$.

However, $\lim\limits_{X\to 1} \overline{E}^{1+}$ is a finite quantity.

$$\pi_\varphi^1 = iXP\int_0^\infty \frac{J_1(\lambda r)\ e^{-\gamma(d-z)}}{(\mu^2\ \gamma+\gamma_E)(\gamma+\gamma_E)}\lambda^2\ d\lambda, \quad (|d|>|z|)\ (z<0). \quad (54)$$

Since $\overline{\pi}^{1-} = 1_\varphi \pi_\varphi^{1-}$ and π_φ^{1-} is not a function of φ, $\overline{E}^{1-} = k_0^2\ 1_\varphi \pi_\varphi^{1-}$. Thus a first order correction term for the reflected field has been found.

The integral (54) must now be evaluated.

Let $\quad d - z = \xi$.

$$Q(\lambda) = \frac{\lambda^2}{(\mu^2\gamma+\gamma_E)(\gamma+\gamma_E)} \quad .$$

Since $Q(\lambda)$ is an even function of λ and

$2J_1(\lambda r) = H_1^{(1)}(\lambda r) + H_1^{(1)}(-\lambda r)$, integral (54) may be written

$$\pi_\varphi^{1-} = \frac{iXP}{2}\int_{-\infty}^{\infty} Q(\lambda)\, H_1^{(1)}(\lambda r)\, e^{-\gamma\xi}\, d\lambda \qquad (55)$$

where $H_1^{(1)}(\lambda r)$ is the Hankel function of the first kind.

The integral is not yet uniquely determined because there are branch points at $\lambda = \pm k_o$ and $\lambda = \pm k$. The integrand is made unique by introducing branch cuts. These cuts and the path of integration are shown in Fig. I under the assumption that $\mu < 1$.

With the plane cut as shown, the poles of the function $Q(\lambda)$ lie on the second Riemann sheet.

In general the integral (55) is difficult to evaluate. Therefore, the problem will be simplified further by assuming:

1) Since the reflected field is of interest when z is comparable to d, and d is very large as compared to a free space wave-length (perturbation theory is only applicable to high frequencies), ξ is large and positive.

2) Since reflections occur in that region of the ionosphere where the electron density is such that $X \doteq 1$, the limit of equation (55) as X tends to unity will be taken. It is assumed that the limit may be

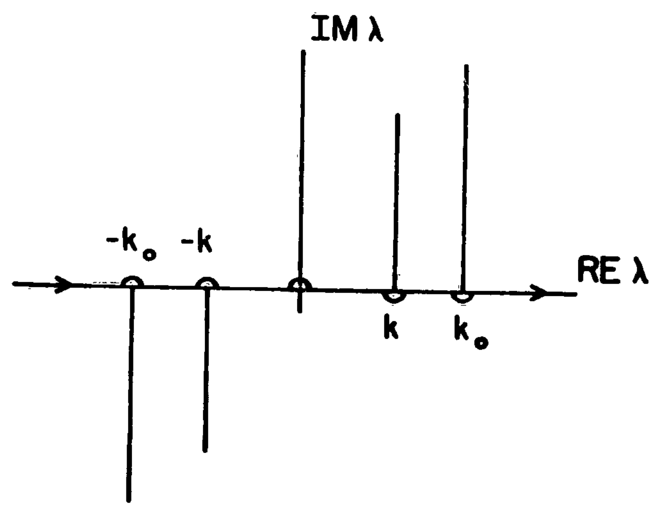

FIGURE I

COMPLEX λ - PLANE

taken inside the integral.

Then equation (55) becomes

$$\pi_\varphi^{1-} = \frac{iP}{2} \int_{-\infty}^{\infty} Q^1(\lambda) H_1^{(1)} (\lambda r)\ e^{-\gamma \xi} d\lambda \qquad (56)$$

where

$$Q^1(\lambda) = \lim_{X \to 1} Q(\lambda) = \frac{\lambda}{\lambda + \gamma} \quad .$$

By using the asymptotic form of the Hankel function and the substitution $\lambda = k_o \sin\theta$, the integral (56) is transformed to

$$\pi_\varphi^{1-} = \frac{Pk_o}{2}\sqrt{\frac{2}{\pi}}\ e^{-\frac{i\pi}{4}} \int_{-\pi/2+i\infty}^{\pi/2-i\infty} \sin\theta \cos\theta\ e^{-i\theta} \frac{e^{ik_o\left[r \sin\theta + \xi\cos\theta\right]}}{\sqrt{k_o\ r \sin\theta}} d\theta \ . \qquad (57)$$

This integral may be approximated by using the saddle point method. To this end, let $r = R \sin\alpha$ and $\xi = R \cos\alpha$ to transform (57) to

$$\pi_\varphi^{1-} = \frac{Pk_o}{2}\sqrt{\frac{2}{\pi}}\ e^{-\frac{i\pi}{4}} \int_C \frac{\sin\theta \cos\theta\ e^{-i\theta}}{\sqrt{k_o r \sin\theta}}\ e^{ik_o R \cos(\theta-\alpha)} d\theta \qquad (58)$$

where the contour C begins at $-\pi/2 + \alpha + i\infty$ and ends at $\pi/2 + \alpha - i\infty$.

Let

$$\varphi(\theta) = ik_o R \cos(\theta - \alpha)$$

$$\varphi'(\theta_S) = -ik_o R \sin(\theta_S - \alpha) = 0 \ .$$

Therefore, there is a saddle point at $\theta = \theta_s = \alpha$.

$$\varphi''(\theta) = -ik_o R \cos(\theta - \alpha) .$$

In the neighborhood of the saddle point

$$\varphi(\theta) = ik_oR - \frac{ik_oR}{2}(\theta - \alpha)^2 \qquad (59)$$

$$(\theta - \alpha) = s\, e^{i\nu}$$

$$\varphi(\theta) = ik_oR - \frac{ik_oR\, s^2}{2}(\cos 2\nu - i \sin 2\nu)$$

$$\operatorname{Im} \varphi(\theta) = k_oR - \frac{k_oR}{2} s^2 \cos 2\nu \quad .$$

The contour C must be deformed into a contour C' which has the property that along C' $\operatorname{Im} \varphi(\theta) = \operatorname{Im} \varphi(\alpha)$. In the neighborhood of the saddle point this condition requires $\nu = \pm \pi/4$. The sign of ν is chosen such that $R_e\varphi(\theta) < 0$ when $s > 0$. In this way it is found that $\nu = -\pi/4$. The contour C' is shown in Fig. II.

From equation (59) equation (58) becomes

$$\pi_\phi^{1-} = \frac{k_oP}{2}\sqrt{\frac{2}{\pi}}\, e^{-\frac{i\pi}{4}} \int_{C'} \frac{\sin\theta \cos\theta\, e^{-i\theta}}{\sqrt{k_or \sin\theta}}\, e^{ik_oR}\, e^{-\frac{k_oR}{2}s^2}\, e^{-\frac{i\pi}{4}} ds \; .$$

Hence, to a good approximation,

$$\pi_\phi^{1-} = \frac{k_oP}{2}\sqrt{\frac{2}{\pi}}\; e^{-\frac{i\pi}{4}}\; \frac{\sin\alpha \cos\alpha\, e^{-i\alpha}}{\sqrt{k_or \sin\alpha}} \int_{-\infty}^{\infty} e^{ik_oR}\, e^{-\frac{k_oR}{2}s^2} ds$$

$$\pi_\phi^{1-} = P\, e^{-\frac{i\pi}{2}} \cos\alpha\, e^{-i\alpha}\, \frac{e^{ik_oR}}{R}$$

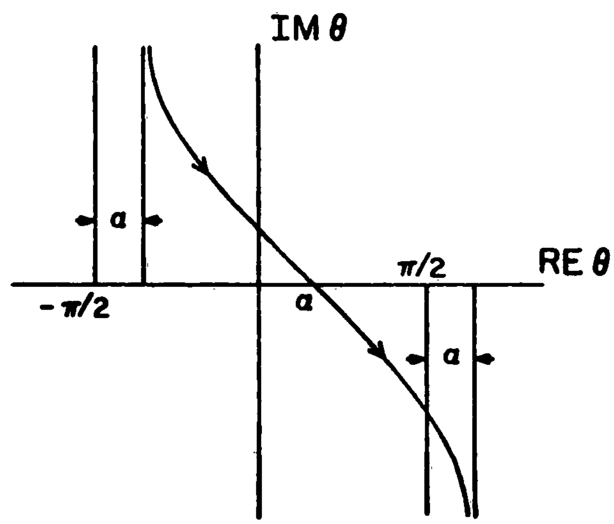

FIGURE II

COMPLEX θ-λ PLANE

The use of the asymptotic form of the Hankel function is justified if $k_o r \sin\alpha \gg 1$. Because of the way in which the angle α was defined, it is seen that π_φ^{1-} takes the form of a spherical wave centered about the dipole image located at $(0, 0, + d)$.

In summary, the expression (60) holds if:

1. $X = 1$
2. $R \gg 1$ and $k_o r \sin\alpha \gg 1$.

Since $\lim\limits_{X\to 1} f^S(\lambda) = -\frac{1}{Y}$

$$\pi^S = - P \frac{e^{ik_oR}}{R} .$$

Therefore, in the region $|d| < |z| \quad z < 0$, the total reflected $\bar{\pi}$ vector may be written

$$\bar{\pi} = - P \left[1_z + 1_\varphi \left| \bar{Y} \right| i \cos\alpha \, e^{-i\alpha} \right] \frac{e^{ik_oR}}{\bar{R}} . \quad (61)$$

The reflected field intensity $\bar{E}$ may be obtained by operating on $\bar{\pi}_R$ by $\nabla(\nabla\cdot) + k_o^2$.

IV. POLARIZATION OF THE $\vec{H}$ VECTOR

The polarization of the reflected field is of some interest. In what follows it proves convenient to introduce standard spherical coordinates. This change is easily made in equation (61) by replacing α by $\pi-\theta$ where θ is the co-latitude. With these changes equation (61) becomes

$$\overline{\pi} = - P\left[1_z + 1_\varphi \left|\overline{Y}\right| e^{i(\theta + \pi/2)} \cos\theta\right] \frac{e^{ik_oR}}{R} . \quad (62)$$

It should be noted that these angles are measured with respect to a coordinate system whose origin is at the image point (0, 0, + d).

Since $1_z = 1_R \cos\theta - 1_\theta \sin\theta$, the components of the reflected $\overline{\pi}$ vector are

$$\pi_R = - P \cos\theta \frac{e^{ik_oR}}{R}$$

$$\pi_\theta = P \sin\theta \frac{e^{ik_oR}}{R}$$

$$\pi_\varphi = - P \left|\overline{Y}\right| e^{i(\theta + \pi/2)} \cos\theta \frac{e^{ik_oR}}{R} .$$

Using the definition of the quantity $P = \frac{i\omega\mu_o}{4\pi k_o^2} I$ and $\frac{k_o^2}{i\omega\mu_o} \nabla \times \overline{\pi} = \overline{H}$ the components of $\overline{H}$ in the radiation region are found to be

$$H_\theta = \frac{ik_oI}{4\pi} \left|\overline{Y}\right| \cos\theta \, e^{i(\theta + \pi/2)} \frac{e^{ik_oR}}{R}$$

$$H_\varphi = \frac{ik_oI}{4\pi} \sin\theta \frac{e^{ik_oR}}{R} .$$

Thus

$$R_eH_\theta e^{-i\omega t} = \frac{k_oI}{4\pi R} |\overline{Y}| \cos\theta \cos(k_oR + \theta + \pi - \omega t)$$

$$R_eH_\varphi e^{-i\omega t} = \frac{k_oI}{4\pi R} \sin\theta \cos(k_oR + \frac{\pi}{2} - \omega t) .$$

Let ψ be the angle between -1_θ and the major axes of the polarization ellipse.

After a simple calculation it is found that[10]

$$\tan 2\psi = - \frac{2 |\overline{Y}| \sin^2\theta \cos\theta}{|\overline{Y}|^2 \cos^2\theta - \sin^2\theta} .$$

V. SUMMARY

It has been shown that by treating the earth's magnetic field of induction as a perturbation, Maxwell's equations and the constitutive relation are simplified. The partial differential equations which determine the n'th coefficient of $|\overline{Y}|^n$ take the form of Maxwell's equations in an isotropic medium. However, the medium is still inhomogeneous. Since, in general, it is not practicable to calculate correction terms of order greater than the first, the frequency must be high.

A first order correction term has been found for the $\overline{H}$ vector reflected from a sharply bounded homogeneous ionosphere. The earth's magnetic field was assumed to be vertical and the source was taken to be a vertical point dipole located in free space a distance d below the interface.

The correction term is expressed in terms of a Fourier-Bessel integral. In order to evaluate this integral, the limit as X tends to unity was taken and the saddle point method was applied. Finally the angle between -1_θ and the major axes of the polarization ellipse of the reflected $\overline{H}$ vector was stated.

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